On the Coefficients of Certain Schlicht Functions.

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In this paper two kinds of schlicht functions will be discussed.

Let us denote by $F(z)$ a function regular in the unit circle and with the expansion

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

about the origin.

§1. It is well known that if $F(z)$ is schlicht and if either it is starlike with respect to the origin or its coefficients are all real, then

$$(1.1) \quad |a_n| \leq n, \quad n \geq 2.$$  

This theorem has been extended by S. Ozaki [1] as follows:

**THEOREM A.** Let $F(z)$ be regular for $|z| < 1$ and starlike in the direction of a diametral line, then $F(z)$ satisfies (1.1).

(In this theorem $F(z)$ is not necessarily schlicht.)

On the other hand it has been shown by M. S. Robertson [2] that if $F(z)$ is schlicht and starlike with respect to a point other than the origin, then

$$|a_2| \leq \cos \left( \frac{\arg F(a)}{a} \right) + \frac{1}{2} \frac{|1 + (a)^2|}{F(a)} \leq 2,$$

$$|a_3| \leq \frac{1}{3} + \frac{2}{3} \cos \left( \frac{\arg F(a)}{a} \right) \left( 1 + \frac{1}{F(a)} \right) \left( 1 + \frac{1}{F(a)} \right) + \frac{1}{6} \frac{|1 + (a)^2|}{a} \leq 3,$$

where $F(a)$ is the star-centre point of the image domain. But he has not touched there the coefficients $a_n$ such that $n \geq 4$.

In this section I shall show by extending Theorem A that the schlicht function starlike with respect to a point other than the origin satisfies (1.1) also for $n \geq 4$.

**DEFINITION 1.** Let $f(z)$ be regular for $|z| < 1$ and let $\xi$ be an arbitrary point such that $|\xi| = 1$, then the straight line $f(\xi), f(-\xi)$ is called a semi-diametral line of $f(z)$.

**REMARK.** When a semi-diametral line passes through the origin, it becomes a diametral line, whose idea was introduced by S. Ozaki [1] and N. G. DeBruijn [4].

**DEFINITION 2.** Let $f(z)$ be regular for $|z| < 1$ and let the image curve of $|z| = 1$ by $f(z)$ be cut by a semi-diametral line at exactly two points, then $f(z)$ is said to be starlike with respect to the semi-diametral line.

**THEOREM 1.** Let $F(z)$ be starlike with respect to a semi-diametral line and let $d$ denote the distance from the origin to the semi-diametral line, then

$$|a_{2n+1}| \leq 2n + 1, \quad n \geq 1,$$

$$|a_{2n}| \leq 2n + |(a_2)^2| + 2, \quad n \geq 1,$$

$$|a_{2n}| \leq 2(1 + d).$$

**PROOF.** Let $F(\xi), F(-\xi)$, $|\xi| = 1$, be the semi-diametral line and let $a = \arg z$. Further let $\beta$ denote the angle formed by the line and the positive imaginary axis, and let $w_o$ be the foot of the perpendicular from the origin to the semi-diametral line.
Then for a sign \( \sigma \) (+ or –) suitably chosen

\[
\Re \sigma e^{i\theta} (F(e^{i\theta}) - w_0) \begin{cases} 
\leq 0 & \text{for } a \leq \theta \leq a + \pi, \\
\geq 0 & \text{for } a + \pi \leq \theta \leq a + 2\pi.
\end{cases}
\]

On the other hand, putting

\[
g(z) = i\tilde{\xi} (\xi - z) (-\xi - z)/z,
\]

we have

\[
\Re \sigma e^{i\theta} = 2 \sin(a - \theta) \begin{cases} 
\leq 0 & \text{for } a \leq \theta \leq a + \pi, \\
\geq 0 & \text{for } a + \pi \leq \theta \leq a + 2\pi,
\end{cases}
\]

which combined with (1.5) gives

\[
\Re \sigma e^{i\theta} (F(z) - w_0) \geq 0, \quad |z| = 1.
\]

Here

\[
g(z) = \sum_{n=1}^{\infty} \left( a_n - \xi^2 a_n \right) z^n + \frac{\xi^2 w_o - \xi (w_o + \xi^2 a_o)}{z}
\]

where \( \xi = \sigma e^{i(\beta - a)} \).

Consequently by Robertson’s lemma [8] we have

\[
\Re \sigma e^{i\theta} \begin{cases} 
\leq 2R(-\xi^2), \\
\leq 2R(-\xi^2), \quad n \geq 1
\end{cases}
\]

From (1.7)

\[
|\xi^2 a_n| \leq 2R(-\xi^2) + |w_o| (|\xi| + |\xi|),
\]

which gives (1.4), since

\[
|\xi| = |\xi| = 1, \quad |w_o| = d, \quad R(-\xi^2) \leq 1.
\]

Next from (1.8) we have

\[
|a_{n+2} | \leq |a_n| + 2.
\]

Putting \( n = 1, 2, \ldots \) in (1.9), we obtain successively

\[
|a_2| \leq |a_1| + 2 \leq 3, \quad |a_3| \leq |a_2| + 2, \quad |a_4| \leq |a_3| + 2 \leq 5, \quad |a_5| \leq |a_4| + 2,
\]

Thus we get (1.2), (1.3).

**Corollary 1.** (Theorem A). Let, in Theorem 1, the semi-diametral line pass through the origin, then \( F(z) \) satisfies (1.1).

**Proof.** Putting \( d = 0 \) in Theorem 1, we get this at once.

**Corollary 2.** Let \( F(z) \) be schlicht in Theorem 1, then \( F(z) \) satisfies (1.1).

**Proof.** Because in this case \( |a_2| \leq 2 \).

**Corollary 3.** Let \( F(z) \) be schlicht and let \( D \) denote the image domain of the unit circle by \( F(z) \). If either (1) \( D \) is starlike with respect to a point which is not necessarily the origin or (2) \( D \) is convex in one direction, then \( F(z) \) satisfies (1.1).

**Proof.** We may assume without loss of generality \( F(z) \) to be regular for \( |z| \leq 1 \).

The case (1). Let \( c \) be the centre of starlikeness of \( D \), and put \( f(z) = F(z) - c \). Then \( f(z) \) has at least one diametral line, which cuts the image curve of \( |z| = 1 \) by \( f(z) \) at exactly two points. This shows that \( F(z) \) has such a semi-diametral
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line passing through the point c as cuts the boundary of D at exactly two points. Accordingly by Corollary 2 \( F(z) \) satisfies (1.1).

The case (2). Let \( l \) denote a straight line which shows the direction of convexity. Then we can easily verify that there exists such a semi-diametral line \( l' \) as is parallel to \( l \) and \( l' \) cuts the boundary of \( D \) exactly two points. Thus this function also satisfies (1.1).

§2. Recently A. Dvoretzky [5] has established the following theorem:

Let \( F(z) \) be schlicht and let \( A(R) \) denote the radius of the largest circle whose centre lies on \( |w|=R \) and whose interior lies in the image of \( |z|<1 \) under \( w=F(z) \). If \( A(R)=O(R^\lambda) \), then

\[
\begin{align*}
(2.1) & \quad |a_n|=O(n\lambda/(2-\lambda)) \quad \text{for } 0<\lambda \leq 1, \\
(2.2) & \quad |a_n|=O(\log n) \quad \text{for } \lambda=0, \\
(2.3) & \quad |a_n|=O(1) \quad \text{for } -2 \leq \lambda < 0, \\
(2.4) & \quad |a_n|=O((n-1/2)/(2-\lambda)) \quad \text{for } \lambda<-2.
\end{align*}
\]

These estimates show a gradual passage from \( |a_n|=O(n) \) for the general schlicht function to \( |a_n|=o(n^{-1/2}) \) for the bounded one.

This theorem has been improved by W. K. Hayman [6] for the two cases \( \lambda=1,0 \) as follows:

(I) When \( \lambda=1 \),

\[
|a_n|=O((n-1/2+K^{c})
\]

on the condition that \( A(R)<cR \) (c<2/3) for \( R>R_0 \), where \( K \) is an absolute constant, and

(II) when \( \lambda=0 \),

\[
|a_n|=O(n^{-1/2} \log n).
\]

(2.5) shows that \( |a_0| \) can be estimated by the same order as (2.4) under a weaker hypothesis on \( A(R) \).

In this section I shall improve the result of Dvoretzky for each value of \( \lambda \) and at the same time show that (2.5) can be made more precise and further the restriction \( c<2/3 \) is unnecessary.

LEMMA. Let \( F(z) \) be schlicht and let \( A(R) \) denote the same quantity as in the above theorem. then

\[
(2.7) \quad |F'(z)| \leq 4A(R)/(1-|z|^2), \quad R=|F(z)|.
\]

PROOF. The function

\[
\zeta=g(z)=\left[ F\left( \frac{z+z_0}{1+z_0z} \right) - F(z_0) \right] / F'(z_0) \quad (1-|z|^2)=z+\cdots, \quad |z_0|<1,
\]

is regular and schlicht in \( |z|<1 \). Moreover \( g(z) \) omits some values on every circle \( |\zeta|=\rho \geq A(R_0)/|F'(z_0)| (1-|z|^2), \quad R_0=|F(z_0)| \), from the definition of \( A(R) \). we have therefore by Koebe's 1/4—theorem

\[
A(R_0)/|F'(z_0)| (1-|z|^2) \geq 1/4,
\]

which gives (2.7).

THEOREM 2. Let \( F(z) \) be schlicht and let \( A(R)=O(R^\lambda) \). Then, (1) in the case \( \lambda=1 \), on the condition that \( A(R)<cR \) for \( R>R_0 \) (where \( c, R_0 \) are constants),
(2. 8) \[ |a_n| = O(n^{-1+2c}) \quad \text{for } \frac{1}{2} < c \leq 1, \]

(2. 9) \[ |a_n| = O(\log n) \quad \text{for } c = \frac{1}{2}, \]

(2. 10) \[ |a_n| = O(1) \quad \text{for } \frac{1}{4} \leq c < \frac{1}{2}, \]

(2. 11) \[ |a_n| = O(n^{-1/2+2c}) \quad \text{for } c < \frac{1}{4}. \]

(2. 12) \[ |a_n| = O(n^{-1/2} \log n)^{1/(1-\lambda)}. \]

**Remark.** We should remark in the case \( \lambda < 1 \) that for the general schlicht function \( F(z) \), \( \lim_{R \to \infty} \frac{A(R)}{R^{\lambda}} \leq 1. \)

**Proof.** Since \( A(R) = O(R^\lambda) \), there are two constants \( c, R_0 \) such that

(1) \[ A(R) < cR^\lambda \quad \text{for } R > R_0, \]

where \( c \) is not necessarily no larger than 1.

From the nature of our problem we may assume that \( F(z) \) is not bounded, and so there is a constant \( r_0 \) \((< 1)\) such that \( \max_{|z|=r_0} |F(z)| = R_0. \)

Now let \( \rho \) \((< 1)\) be a number larger than \( r_0 \), then we can find a real \( a \), depending on \( \rho \), such that \( |F(\rho e^{ia})| = \max_{|z|=\rho} |F(z)| > R_0. \)

Putting \( |F(\rho e^{ia})| = R(t) \), we have

(2) \[ \frac{dR(t)}{dt} \leq |F'(t e^{ia})|. \]

Take \( \rho_0 \) such that \( |F(\rho_0 e^{ia})| = R_0 \) besides \( |F(\rho e^{ia})| > R_0 \) for all \( t \) in \( \rho_0 < t < \rho \), then by (2. 7), (1), and (2) we find

(3) \[ \int_{\rho_0}^{\rho} \frac{dR(t)}{R^\lambda} < \int_{\rho_0}^{\rho} \frac{4c}{1-t^2} dt. \]

Hence

(4) \[ R(\rho) < R_0 \left[ (1+\rho)/(1-\rho) \right]^{2c}, \quad r_0 < \rho < 1. \]

Consequently by the area-principle we have

(5) \[ \sum_{n=1}^{\infty} n |a_n|^2 \rho^{2n} \leq \left( \max_{|z|=\rho} |F(z)| \right)^2 < R_0 \left[ \frac{1+\rho}{1-\rho} \right]^{4c}. \]

Hence

(6) \[ |a_n| < R_0 \rho^{-1/2} \rho^{n \left[ \frac{1+\rho}{1-\rho} \right]^{2c}}, \quad n \geq 2, \quad 0 < \rho < 1. \]
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Put here \( p = \frac{n}{n+1} \), then

\[
|a_n| < R_0 n^{-1/2} \left(1 + \frac{1}{n}\right)^n (2n+1)^{2c} < eR_0 n^{-1/2} (2n+1)^{2c}, \; n \geq 2,
\]

which proves that (2.11) holds for an arbitrary non-negative real number \( c \).

Next we shall show that (2.11) can be made more precise for \( 1/4 \leq c \leq 1 \). Set

\[
G(z) = \sqrt{F(z^2)} = z_1 \sqrt{F(z^2)/z^2} = z + \ldots,
\]

then \( G(z) \) is regular and schlicht for \(|z| < 1\), and further we have

\[
(6) \quad \int_0^{2\pi} |F(re^{i\theta})| \, d\theta = \int_0^{2\pi} |G(re^{3i\theta})| \, d\theta = \int_0^{2\pi} |G(r^{1/2} e^{i\theta})| \, d\theta.
\]

Let \( J(r) \) denote the area of the image of \(|z| < r < 1\) by \( G(z) \), then from (4) we have

\[
J(r) \leq \pi \max_{|z| = r} |G(z)|^2 \leq \pi \max_{|z| = r} |F(z^2)| \leq \pi R_0 \left(\frac{1 + r^2}{1 - r^2}\right)^{2c} \leq \pi R_0 \left(\frac{1}{1 - r}\right)^{2c}.
\]

On the other hand by the distortion-theorem concerning schlicht functions we have

\[
J(r) \leq \pi \max_{|z| = r} |F(z^2)| \leq \pi r^2/(1 - r^2)^2.
\]

Consequently for \( 1/4 \leq \rho < 1 \),

\[
\frac{1}{2\pi} \int_0^{2\pi} G(\rho e^{i\theta}) \, d\theta = \frac{\rho}{2\pi} \int_0^{2\pi} J(r) \, dr < \frac{1}{4} \int_0^{2\pi} \frac{2r}{(1 - r^2)^2} \, dr + 2R_0 \left(\frac{1}{1 - r}\right)^{2c} dr
\]

\[
(7) \quad \begin{cases}
A_1 \left(\frac{1}{1 - \rho}\right)^{1+2c} & \text{for } c > \frac{1}{2}, \\
A_2 \log \frac{1}{1 - \rho} & \text{for } c = \frac{1}{2}, \\
A_3 & \text{for } c < \frac{1}{2},
\end{cases}
\]

where \( A_1, A_3 \) are constants depending only upon \( R_0 \) and \( c \), and \( A_2 \) is a constant depending only upon \( R_0 \).

We have now by Cauchy’s theorem

\[
|a_n| = \left| \frac{1}{2\pi i} \int_{|z| = \rho} \frac{F(z)}{z^{n+1}} \, dz \right| \leq \frac{1}{2\pi \rho^n} \int_0^{2\pi} |F(\rho e^{i\theta})| \, d\theta,
\]

and it follows from (6) and (7) that for \( n \geq 2 \), \( 1/4 \leq \rho < 1 \),

\[
|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} G(\rho^{1/2} e^{i\theta}) |^2 d\theta < \begin{cases}
A_1 \rho^n \left(\frac{1}{1 - \sqrt{\rho}}\right)^{-1+2c} & \text{for } c > \frac{1}{2}, \\
A_2 \rho^n \log \frac{1}{1 - \sqrt{\rho}} & \text{for } c = \frac{1}{2}, \\
A_3 \rho^n & \text{for } c < \frac{1}{2},
\end{cases}
\]

Putting here \( \rho = \left(1 - \frac{1}{n}\right)^2 \), we obtain for \( n \geq 2 \),
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\[
|a_n| < \begin{cases} 
4e^2A_1 n^{-1+2c} & \text{for } c > \frac{1}{2}, \\
4e^2A_2 \log n & \text{for } c = \frac{1}{2}, \\
4e^2A_3 & \text{for } c < \frac{1}{2},
\end{cases}
\]

from which (2.8), (2.9), and (2.10) follow.

I. When \( \lambda < 1 \), from (3)

\[
R(\rho) < (R_0^{1-\lambda} + 2c \log \frac{2}{1-\rho})^{1/(1-\lambda)}, \quad r_0 < \rho < 1.
\]

Hence \( \max_{|z|=\rho} |F(z)| = R(\rho) < A_4 \left( \log \frac{1}{1-\rho} \right)^{1/(1-\lambda)} \) for every \( \rho \) in \( \frac{1}{2} \leq \rho < 1 \),

where \( A_4 \) is a constant depending only upon \( R_0 \) and \( c \).

Therefore by the area-principle we have

\[
\sum_{n=1}^{\infty} n^2 |a_n|^2 \rho^{2n} < A_4 \left( \log \frac{1}{1-\rho} \right)^{1/(1-\lambda)}.
\]

Hence

\[
|a_n| < A_4 n^{-1/2} \rho^{-n} \left( \log \frac{1}{1-\rho} \right)^{1/(1-\lambda)}, \quad n \geq 2, \quad \frac{1}{2} \leq \rho < 1.
\]

Putting here \( \rho = 1 - \frac{1}{n} (n \geq \frac{1}{2}) \), we obtain (2.12).

We thus complete the proof of the theorem.

References.